The Well Tempered Pythagorean: The Remarkable Relation Between Western and Natural Harmonic Music

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Abstract

True natural harmony, popularized by sixth century BC Greek philosopher Pythagoras, required simultaneously sounded frequency to have ratios equal to the ratios of small whole numbers. The tempered scale of western music, on the other hand, requires all semitone frequencies to have a ratio of the awkward number, $\sqrt[12]{2}$. The number $\sqrt[12]{2}$ and, except for the octave, all western music intervals, can't be exactly expressed as the ratio of small numbers. Indeed, the frequency ratios can't be exactly expressed as the ratio of *any* whole numbers. They are irrational. Nevertheless, the tempered scale can be used to generate natural Pythagorean ratios to an accuracy often audibly indistinguishable to a trained listener. The tempered scale, in addition, allows freedom of key changes (modulation) in musical works difficult to instrumentally achieve in the Pythagorean system of harmony. The geometrically spaced frequency intervals of the tempered scale also make perfect use of the logarithmic frequency perception of the human ear. The ability of the tempered scale to generate near perfect Pythagorean harmonies can be demonstrated in a number of ways. A deeper question asks why this remarkable relationship exists. There is, remarkably, no fundamental mathematical or physical truth explaining the observed properties. The philosophical answer, we propose, is either a wonderful cosmic coincidence - or the clever design of a Creator desiring our appreciation of both the intricate and beautiful harmonies of western music and the flexibility to richly manipulate them in almost innumerable ways.

Introduction

Newtonian physics applied to the vibrating strings of violins, violas and guitars; and the vibrating air columns of bugles, clarinets, trombones and pipe organs, show that available tones of a string or an air column of fixed length are related by integer multiples (*harmonics*) of a fundamental frequency. Tonal color is largely crafted by choice of the strength of the components of the harmonics in a tone.

The physics of vibrating strings and air columns can be used to generate a partial differential equation dubbed *the wave equation*. When subjected to boundary conditions (*e.g.* a string is constrained not to move at the bridge boundary of a guitar), the wave equation gives rise to a harmonic or *Fourier series* solution of the wave equation corresponding to integer multiples of the lowest frequency allowed by the boundary constraints.[1]

The Mth harmonic of a reference tone, or *root*, is simply M+1 times the frequency of the reference frequency and can be expressed as the ratio 1:M+1. For a vibrating string or air column, the root can be taken as the lowest frequency allowed by the physics of the imposed boundary constraints. Simultaneous sounding of notes corresponding to the root and its first five harmonics constitute a natural major chord. Clearly, then, notes with frequency ratios of 1:M+1 form pleasing harmony when M is small.

More generally, Pythagorean harmony claims two audio tones will harmonize when the ratio of their frequencies are equal to a ratio of small whole numbers (*i.e.* positive integers). This follows directly from the pleasing harmony

[1] Mathematical details of the derivation of the wave equation for a vibrating string and its harmonic (or Fourier series) solution is in Appendix A.

of harmonics. For example, the second harmonic (1:3) and the third (1:4) harmonic harmonize. These two notes have a relative ratio[2] of 3:4 and, indeed, form a the interval of a natural perfect fourth. This pleasing musical interval is, as required by Pythagoras, the ratio of two small whole numbers.

Western music, on the other hand, is not based on the ratio of small whole numbers. It is, rather, built around twelfth root of two = $\sqrt[12]{2} = 2^{\frac{1}{12}} = 1.05946...$. The number 12 comes from the division of octave into 12 equally spaced chromatic steps. By definition, $\sqrt[12]{2}$, when multiplied by itself 12 times, is equal to two.[3] The ratio of the frequency of any two adjacent semitones[4], such as C# to C, is $\sqrt[12]{2}$. Since a perfect fourth is five semitones, the ratio of frequencies is $\left(\sqrt[12]{2}\right)^5$. This seems a far cry from the Pythagorean frequency ratio of 3:4. When, however, the numbers are evaluated, we find

$$\frac{4}{3} = 1.33333 \approx \left(\sqrt[12]{2}\right)^5 = 1.33484. \qquad (1)$$

The difference between these notes is a miniscule audibly indistinguishable[5] 1.955% of a semitone, or 1.955 cents.[6]

The near numerical equivalence in Equation (1) illustrates an more general awesome property of the tempered scale of western music. Using the awkward number, $\sqrt[12]{2}$, frequencies can be generated whose ratios are nearly identical to ratios of whole numbers. The tempered scale also offers important advantages in comparison to a strict Pythagorean scale.

- **Modulation.** The tempered scale allows modulation among different musical keys. The tonic or reference note can be changed since, in the tempered scale, no note is favored above another. Using the tempered scale, for example, one can change melody and harmony from the key of C to the key of G using the same set of notes. This cannot be done in the Pythagorean system. This remarkable property of the tempered scale was celebrated by J. S. Bach in *The Well Tempered Clavier* wherein all twenty four major and minor scales were used in a single work.
- **Dynamic Range Perception.** The human sense of frequency perception is approximately logarithmic. This allows a larger dynamic range of perception.[7] The tempered scale logarithmically divides each octave nicely into twelve equal logarithmic intervals.

Why does the flexible tempered scale so well mimic the structured whole number Pythagorean frequency ratios? The answer, as we illustrate, is either a marvelous coincidence of nature or clever design of the Creator of music. There is no foundational mathematical or physical reason the relationship between Pythagorean and tempered western music should exist. But is does. The rich flexibility of the tempered scale and the wonderful and bountiful archives of western music are testimony to this mysterious relationship.

Pythagorean Harmony

Harmony in western music is based on *harmonics* - also called overtones. According to Pythagoras, tones are harmonious when their frequencies are related by ratios of small whole numbers. The interval of an octave, or *diapason*, is characterized by ratios of 1:2, 2:1

[2] Equal to the ratio of one fifth to one fourth.

[3] Another illustration is compound interest. A one time investment made at an annual rate of interest of $\sqrt[12]{2}$ - 1 = 5.946… % annual interest will double in 12 years.

[4] A piano tuner will tell you, as is the case with most quantifiable human characteristics, the ratio is not exact. Lower notes on a piano, for example, are tuned a bit lower than dictated by the $\sqrt[12]{2}$ ratio.

[5] The method to compute the semitones or cents between two frequencies is discussed in Appendix B.

[6] A semitone is divided into 100 cents.

[7] Human perception of sound intensity is also roughly logarithmic. If were not, the sound of 100 violins would be perceived as 100 times as loud as that of one. Thankfully this is not the case and we can enjoy in our comfortable hearing range a violin solo or an orchestra of violins. The increase in volume from 1 to 10 violins results roughly in the same incremental volume increase as 10 to 100 violins.

and 2:4. If any frequency is multiplied by 2^N where N is an integer, the resulting frequency is related to the original frequency by N octaves. For example, A above middle C is currently, by universal agreement, is 440 Hz[8]. Then 440 Hz ÷ 2 = 220 Hz is A below middle C and $440\times2^3 = 440\times8 = 3520$ Hz is the frequency of A a total of N=3 octaves above A above middle C. The ratio of 2:3 is the perfect fifth or *diapente* interval while 3:4 is the perfect fourth or *diatesseron* interval.

Numbers that can be expressed as ratio of integers are *rational* numbers. Pythagoras claims harmony occurs between notes when the ratio of their frequencies are small whole numbers.

Harmonics result naturally from the physics of vibrating strings and air columns. The rules for the first few harmonics follow. The intervals cited are Pythagorean (or *natural*) since there relations are determined by ratios of small numbers. After each entry is the note corresponding to a root of middle C, denoted C4. The closest tempered notes are shown in Figure **Error! Reference source not found.** when middle C is the root.

0. 1:1 defines the reference note or root. C4.
1. 2:1, with twice the frequency, is an octave above the root. C5.
2. 3:1 is the perfect fifth (2:1) of the first (2:1) harmonic. G5
3. 4:1 is twice the frequency of the first harmonic and is therefore two octaves above the root. C6
4. 5:1 is the major third of the third (4:1) harmonic. E6.
5. 6:1 is twice the frequency of the second (3:1) harmonic and is therefore the perfect fifth of the third (4:1) harmonic. G6.
6. 7:1 is the minor seventh of the third harmonic. Bb6.
7. 8:1 is three octaves above the root. C7.
8. 9:1 is the third harmonic of the second harmonic. It is therefore the major second of the seventh harmonic. (D7)
9. 10:1 is an octave above the fourth harmonic, and therefore the major third of the seventh harmonic. (E7).
10. 11:1 (F#7)
11. 12:1 is an octave above the 5^{th} harmonic. (G7)
12. 13:1 (Ab7)
13. 14:1 is an octave above the sixth harmonic. (Bb7)
14. 15:1 is both the second harmonic of fourth harmonic and the fourth harmonic of the second harmonic. (B7)
15. 16:1 is an octave above the seventh harmonic (C8).

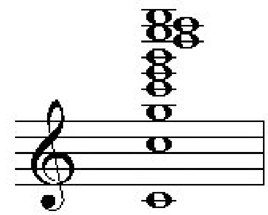


Figure 1: Tempered notes closest to eight harmonics when the root is middle C (a.k.a. C4). The first four harmonics are numbered. Harmonics 2 through 4 (G4,C5,E5,G5) are those used in bugle melodies.

There exist variations of harmonic frequencies from the corresponding tempered notes. This variation is shown in Table 1. The fourth column, headed **Ratio**, contains the normalized frequencies of the harmonics normalized (divided) by the frequency of the root. The next column, labeled **Temper**, contains the normalized frequency of the corresponding tempered note. The **Cents** column contains the error between the harmonic and its tempered equivalent. The error for the lower harmonics is small, as is the error between harmonics whose ratios are products of small numbers, *e.g.* the fifth harmonic where $6 = 3\times2$ has an error of only 2 cents and the fifteenth harmonic, with ratio $16 = 2^4$, has no error whatsoever. Harmonics with larger prime ratios, *e.g.* the sixth harmonic with ratio 7, tend to have larger errors. Indeed, the 10^{th} harmonic corresponding to the prime ratio 11:1, has an error of 48.7 cents, less than 2 cents from being closer to an F than an F#. The 16^{th} harmonic, with prime ratio 1:17, is so far removed from the C4 root it has lost most if not all of its musical harmonic relationship. The 17^{th} harmonic, of the other hand, has a ratio of 18 and is exactly an octave above the eighth harmonic.

Note all tempered notes with the same name deviate from the Pythagorean frequency by same amount. All C's have an error of 0 cents and thus have no deviation. All G's have a deviation of –2 cents, all E's 13.7 cents, etc.

[8] Hz = hertz = cycles per second.

The sixth harmonic deviates over 32 cents from the corresponding tempered note. This difference is enough to be detected by the untrained musical ear.

Harmonic	Note	ST's	Ratio	Temper	cents
0	C4	0	1	1.000	0.0
1	C5	12	2	2.000	0.0
2	G5	19	3	2.997	-2.0
3	C6	24	4	4.000	0.0
4	E6	28	5	5.040	13.7
5	G6	31	6	5.993	-2.0
6	Bb6	34	7	7.127	31.2
7	C7	36	8	8.000	0.0
8	D7	38	9	8.980	-3.9
9	E7	40	10	10.079	13.7
10	F#7	42	11	11.314	48.7
11	G7	43	12	11.986	-2.0
12	Ab7	44	13	12.699	-40.5
13	Bb7	46	14	14.254	31.2
14	B7	47	15	15.102	11.7
15	C8	48	16	16.000	0.0

Table 1: The first fifteen **harmonics** of the root C4. The tempered note closest to the harmonic is shown. The column **ST** refers to the number of tempered semitones from the root. The **ratio** column lists the integer multiple of the root frequency. The **tempered** column, to be compared to the ratio column, is the frequency ratio when the western tempered scale is used. **Cents** is the error between the Ratio and Tempered frequencies. One hundred cents is a semitone.

The first four harmonics with the root form a natural major chord, *e.g.* C C G C E. Removing the redundant notes leaves C G and E. Including the next two harmonics yields a seventh chord (C G E Bb) and three more harmonics a ninth chord (C G E Bb D). We can continue to an eleventh (C G E Bb D F) and thirteenth (C G E Bb D F A) chord. All of the numerical chord names are odd numbers simply because even numbers reflect of an octave relationship and add no new notes to the chord. The 6^{th} harmonic, with ratio *seven*, is, indeed, the new note in the *seventh* chord (Bb). Likewise, the 8^{th} harmonic with ratio *nine*, is added to obtain the *ninth* chord. Continuing further gives deviation. The 10^{th} harmonic (F#) with ratio *eleven* is different from new note in an *eleventh* chord (F). As noted, though, this harmonic is less than two cents from being closer to F than F#. Similarly, the thirteenth (A), build by adding a major third to the eleventh, deviates from the note closest to *eleven* times the frequency of the root (Ab) which corresponds, instead, more closely to the addition of a minor third.

Melodies of Harmonics: Bugle Tunes[9]

Harmonics with ratios of 1:3 through 1:6 are used in the melodies played by bugles, including *Taps* played at military funerals and *Revelry* played to wake soldiers in the morning. The simple four note music for Taps is shown in Figure 2. *Revelry,* played with the same four notes, is shown in Figure 3.

The bugle, when unwrapped, is a simple vibrating air column as illustrated in Figure 4. The vibrating lips of the bugle player determine the vibration *mode* of the air. The vibration modes shown are those used in bugle melodies. The bugle therefore sounds true Pythagorean harmonics and not tempered note intervals. The mode with three "bumps" at the top of Figure 4 is the second harmonic and corresponds the lowest of the four notes. The vibration mode with six "bumps" is the highest.

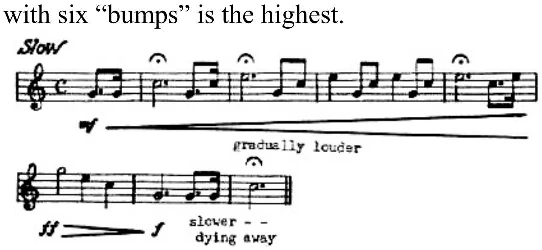


Figure 2: *Taps* is played with four notes. They are the 3^{rd} through 6^{th} harmonics of the bugle's vibrating air column.

Pythagorean and Tempered String Vibrations

A guitar string vibrates similarly to the air column patterns in Figure 4. We assume throughout that, as is the case for a guitar or violin, a string has uniform linear mass density (*i.e.* it is not skinny in some places and fat in others) and is under a constant tension. Different

[9] The music files of bugle melodies are from http://www.computingcorner.com/holidays/vets/taps.html wherein much appreciated permission for educational use is granted.

notes are sounded only be changing the effective length of the string or, as is the case with generation of harmonics, applying initial conditions that prompt the string not to move at one or more points.

The vibration modes of a string are shown in Figure 5 Lightly touching the middle of the string over the twelfth fret bar and plucking results in the string vibrating in two halves. The sounded tone is an octave higher than the string played open. Vibration in two sections continues even after the finger is removed from the string. This is the first harmonic of the note sounded by the open string.

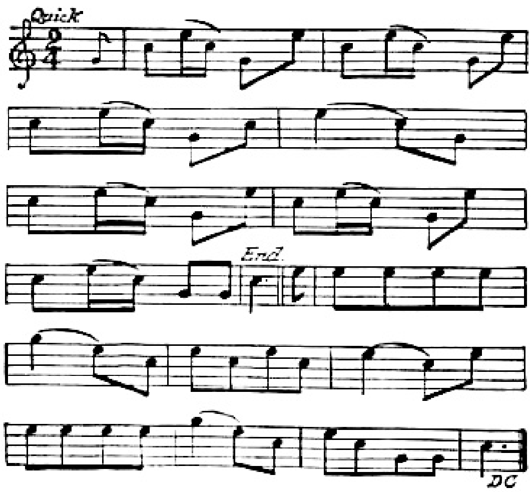


Figure 3: Like *Taps, Revelry* is played with four notes. They are the 2rd through 5th harmonics of the bugle's vibrating air column.

The note sounded in the first harmonic is equivalent to the note sounded by a string of half the length. In other words, the first harmonic in Figure 6 can be viewed as two independent strings vibrating, each string being half the length of the open guitar string. The twelfth fret bar divides the string into two equal pieces. If the string is depressed on the twelfth fret in a conventional matter, the sounded note, after plucking, is the same as the note of the string's first harmonic.

Placing the finger lightly over the seventh fret bar and plucking results in the string vibrating in three equal pieces. (See Figure 5.) This is the second harmonic. Besides the bridges that mechanically constrain the string from vibrating, there are two nodes where the string is not moving. One is over the seventh fret bar. While the string is vibrating, this point on the string can be lightly touched and the string continues to vibrate. The same is true if the string is lightly touched over the 19th fret bar. As is illustrated in Figure 5, this is one of two places, other than the bridges, where the string does not vibrate. Placing the finger on the twelfth fret, on the other hand, interupts the vibration of the string and the note ceases to sound.

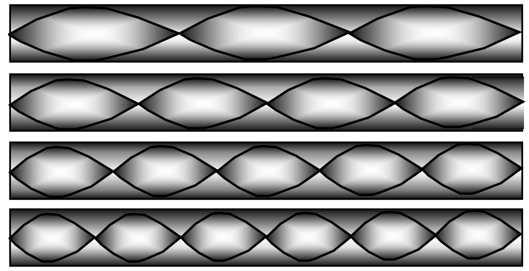


Figure 4: Vibration modes in an air column for the second (top) through fifth (bottom) harmonics. These modes form the four notes for all bugle melodies including *Taps* and *Revelry*.

The note sounded in the second harmonic can be viewed as resulting from the vibration of three independent strings each with a length of one third that of the open string. Depressing the nineteenth fret, which leaves one third of the length on the business end of the string, yields this same note, when plucked, as the second harmonic.

Continuing,, a finger lightly touching the string over the fifth fret bar will, after plucking, result in the string vibrating in four equal parts. This mode corresponds to the vibration shown second from the top in Figure 4 and sounds the third harmonic of the open string's root note. Similarly, lightly touching above the fourth fret gives the fourth harmonic and the third fret gives the fifth.

Note that, by moving the finger between the seventh, fifth, fourth and third fret, the sounded notes are those necessary to play the bugle tunes in Figures 2 and 3. Like the vibrating air column, these harmonics can be sounded by applying the proper initial conditions and stimuli. For the bugle, the stimulus is air and the boundary conditions the frequency of the bugler's vibrating lips. For the string, the boundary conditions are imposed through lightly touching the string while the stimulus is a simple pluck. Remarkably, these physically different systems with different physics display similar musical (and physical) properties. Each obeys the wave equation – a partial differential equation imposed by fundamental Newtonian physics. The wave equation is predominant in physics and is applicable to numerous areas. Besides vibrating strings and air columns, the wave equation can be derived for phenomena ranging from heat transfer to the propagation of light rays. The wave equation is derived in Appendix A for the case of a vibrating string.

Fret Bar Calibration: The Tempered Scale from the Perspective of the Vibrating String

The location of the mode nodes at fret bars in Figure 5 is manifest from the same condition that allows natural Pythagorean music to be approximated by the tempered scale. As noted, a string in Figure 5 with one third the length of the open string, and under the same tension, will sound a note equal to the second harmonic. The string is one third the length and sounds three times the frequency. The relation is the same with all of the harmonics. The fourth harmonic can be sounded with a string of one fifth the length. The frequency is five times higher. We conclude, then, that for a string under a given tension, the string's length is inversely proportional to its frequency. If f is the frequency in Hertz, and ℓ is the length, then

$$f \times \ell = v \tag{2}$$

where, for a given string under constant tension, v is the constant of proportionality.[10] This relation can be derived more forthrightly using the mathematics of physics.

Using Equations 1 and 2, the locations of the fret bars can be obtained. If an open string of length $\ell = \ell_0$ sounds a frequency of f_0, the same string, shortened to length ℓ_1 by the first fret bar as illustrated in Figure 6, should have a frequency equal to a semitone above f_0. This frequency is $f_1 = \sqrt[12]{2} f_0$. From Equation (2), we know that $f_0 \times \ell_0 = f_1 \times \ell_1$. Substituting $f_1 = \sqrt[12]{2} f_0$ and solving gives $\ell_1 = \ell_0 / \sqrt[12]{2} = 0.943874 \times \ell_0$.

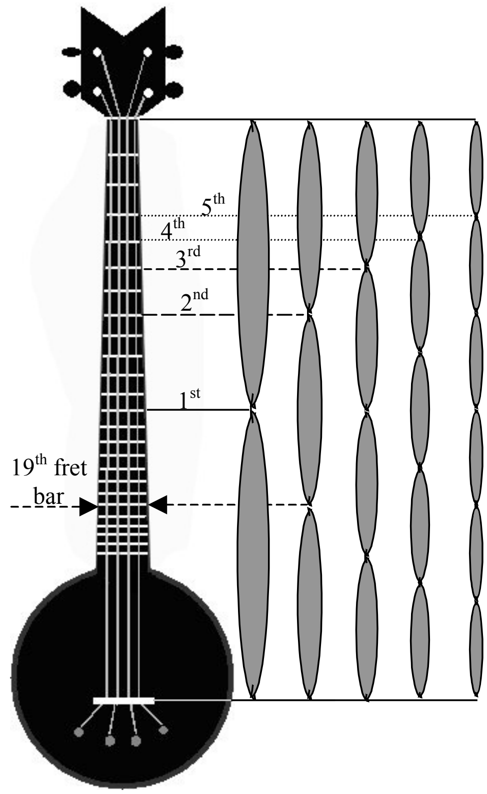


Figure 5: Vibrating strings can, similar to vibrating air columns, generate harmonics. On a guitar, lightly touching at the 12th fret and plucking results in the string vibrating in two pieces and sounds the first harmonic of the root tone of the open string. Lightly touching the seventh fret and plucking results in three vibrating string sections and the second harmonic. The process can be repeated, as shown here, for higher order harmonics. The volume of the tone obtained, however, diminishes as the harmonic number increases.

The length ℓ_2 can likewise be determined by requiring the sounded frequency be a semitone above f_1. Following the same procedure, we find $\ell_2 = \ell_1 / \sqrt[12]{2} = \ell_1 / \left(\sqrt[12]{2}\right)^2$. Continuing this induction, the distance, ℓ_n, between the bridge and the nth fret bar is

$$\ell_n = \frac{\ell}{\left(\sqrt[12]{2}\right)^n} = \ell \times 2^{\frac{-n}{12}} \tag{3}$$

Here are some interesting tempered lengths that lie close to the natural or Pythagorean harmonics.

[10] Indeed, v is the fixed velocity of the wave on the string. If ℓ is in feet and f is in Hertz, then v has units of feet per second. The velocity is related to the string parameters by $v = \sqrt{T/\rho}$ where T is the string's tension (a force) and ρ is the string's linear mass density (mass per unit length). See Appendix A for more details.

- The n = twelfth fret bar is at $\ell_{12} = \ell / 2 =$ half of string's length.
- The $n = 7^{\text{th}}$ fret bar, constituting an interval of a perfect fifth, is at $\ell_7 = \ell / \left(\sqrt[12]{2}\right)^7 = 0.66742 \times \ell \approx \frac{2}{3}\ell = 0.66667 \times \ell.$
- The $n = 5^{\text{th}}$ fret bar, constituting an interval of a perfect fourth, is at $\ell_7 = \ell / \left(\sqrt[12]{2}\right)^7 = 0.66742 \times \ell \approx \frac{2}{3}\ell = 0.66667 \times \ell.$
- The $n = 19^{\text{th}}$ fret bar (see Figure 6) roughly should leave one third of the string's length. $\ell_{19} = \ell / \left(\sqrt[12]{2}\right)^{12} = 0.33371 \times \ell \approx \frac{1}{3}\ell = 0.33333 \times \ell.$

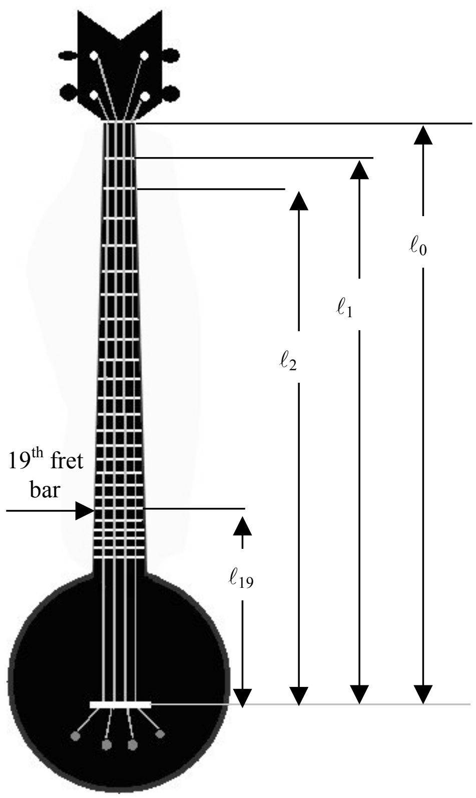


Figure 6: The length of a string from bridge to bridge is $\ell = \ell_0$. The length, ℓ_n, from the bridge to the nth fret bar is

$$\ell_n = \ell / \left(\sqrt[12]{2}\right)^n .$$

Table 2 contains a more complete listing of the harmonic lengths of the vibrating string and the corresponding tempered scale. The **fret** column refers to the string length when depressed at the nth fret. The **temper** column contains the length from the nth fret bar to the bridge. Pythagorean lengths, like Pythagorean frequencies, Since, from Equation 2,

$$\frac{f_n}{f_0} = \frac{\ell_0}{\ell_n},$$

we deduce that, if harmonious Pythagorean frequency ratios require expression as the ratio of small whole numbers, so must the length of strings producing notes of harmony. Thus, good harmony requires

$$\frac{\ell_n}{\ell_0} = \frac{\text{num}}{\text{den}} = \text{ratio}$$

where the numerator, num, and the denominator, den, are small whole numbers. These small whole numbers are shown in the columns in Figure 6. The error between the tempered string length and the natural Pythagorean string length, shown in the column **ratio**, is shown in Table 2 in the **cents** column. Not included are the major and minor second, major seventh, and the tritone. Each is dissonant with respect to the root and defeats the purpose of tabulating the comparison between harmonious intervals.

fret	interval	temper	num	den	ratio	cents
0	**root**	**1.000**	**1**	**1**	**1.000**	**0**
1	m 2nd	0.944				
2	M 2nd	0.891	7	8	0.875	31
3	**m 3rd**	**0.841**	**5**	**6**	**0.833**	**16**
4	**M 3rd**	**0.794**	**4**	**5**	**0.800**	**-14**
5	**P 4th**	**0.749**	**3**	**4**	**0.750**	**-2**
6	**tritone**	**0.707**				
7	**P 5th**	**0.667**	**2**	**3**	**0.667**	**2**
8	**m 6th**	**0.630**	**5**	**8**	**0.625**	**14**
9	**M 6th**	**0.595**	**3**	**5**	**0.600**	**-16**
10	m 7th	0.561	4	7	0.571	-31
11	M 7th	0.530				
12	**octave**	**0.500**	**1**	**2**	**0.500**	**0**

Table 2: Tempered guitar fret bar spacing versus string length based on natural Pythagorean intervals. The prefix "m" denotes "minor", "M" major, and "P" perfect.

An inverted interval pair adds to an octave. An example is the minor third and the major sixth. Note, in table 2, the cents error for inverted interval pairs adds to zero.

We could, if desired, continue Table 2 to include frets above twelve. Doing so, however, is uninteresting. The entries in the **temper** column will be halved. Fret 13, for example, will contain the entry of half of fret 1, equal to 9.944/2 = 4.872. Fret 14 in the **temper** column will contain half the value of that in fret 2. Fret 112 will contain half the value in fret 100, etc. The **num** column, on the other hand, remains the same for the next octave. Each entry in the den column, on the other hand, is divided by 2 for the next octave, $2^2=4$ for the following octave, etc. Thus, for the immediate next octave (frets 13 though 24), the entries in the **ratio** column are half of what they in the table entry shown. In the next highest octave (frets 25 through 36), the entries will be divided by $2^2=4$, etc. The result is that the entries in the **cents** column remain the same for every octave.

Den / Num	1	2	3	4	5	6	7	8	9
1	1.00	0.50	0.33	0.25	0.20	0.17	0.14	0.13	0.11
2	2.00	1.00	0.67	0.50	0.40	0.33	0.29	0.25	0.22
3	3.00	1.50	1.00	0.75	0.60	0.50	0.43	0.38	0.33
4	4.00	2.00	1.33	1.00	0.80	0.67	0.57	0.50	0.44
5	5.00	2.50	1.67	1.25	1.00	0.83	0.71	0.63	0.56
6	6.00	3.00	2.00	1.50	1.20	1.00	0.86	0.75	0.67
7	7.00	3.50	2.33	1.75	1.40	1.17	1.00	0.88	0.78
8	8.00	4.00	2.67	2.00	1.60	1.33	1.14	1.00	0.89
9	9.00	4.50	3.00	2.25	1.80	1.50	1.29	1.13	1.00

Table 3: A table of the ratio of small numbers.

Tempering Pythagoras

By example, we have demonstrated western music's tempered scale, based on the awkward irrational number, $\sqrt[12]{2}$, relates closely to the Pythagorean harmonies given by the ratios of small numbers. Where does this strange number $\sqrt[12]{2}$ come from and why does it mimic lower ordered natural harmonics so well? The number, $\sqrt[12]{2}$, cannot be expressed as the ratio of whole numbers and is therefore an irrational number.[11]

This begs the question. Why does the tempered scale so closely approximate the natural Pythagorean small number ratios. Remarkably, there is no mathematical or worldly philosophical answer. The relationship is either a remarkably fortuitous accident of nature or divine design that allows the Pythagorean system of harmony to be characterized by the tempered scale.

The tempered scale arises through numerous directions of analysis of Pythagorean harmony. Examples have been given for the case of harmonics and guitar fret calibration. Some more directed numerical inspection follows.

A Literal Interpretation of Pythagoras' Lemma

Let's look at the Pythagorean claim literally: frequencies reducible to ratios of small numbers are harmonically pleasing. For "small numbers", let's choose the integers one through nine. This is shown in Table 3. The numerator, denoted **Num**, is in the column at the left and the denominator, **Den**, is in the top row. The ratio **Num/Den** is entered in the table. According to Pythagoras, all of these ratios should harmonize.

	1	2	3	4	5	6	7	8	9
1	C 4	C 3	F 2	C 2	Ab 1	F 1	D 1	C 1	Bb 0
2	C 5	C 4	F 3	C 3	Ab 2	F 2	D 2	C 2	Bb 1
3	G 5	G 4	C 4	G 3	Eb 3	C 3	A 2	G 2	F 2
4	C 6	C 5	F 4	C 4	**Ab 3**	F 3	D 3	C 3	Bb 2
5	E 6	E 5	A 4	E 4	C 3	**A 3**	**F# 3**	E 3	D 2
6	G 6	G 5	C 5	G 4	Eb 4	C 4	**A 3**	G 3	F 2
7	Bb 6	Bb 5	Eb 5	**Bb 4**	**F# 4**	Eb 4	C 4	**Bb 3**	**Ab 3**
8	C 7	C 6	F 5	C 5	Ab 4	F 4	D 4	C 4	**Bb 3**
9	D 7	D 6	G 5	D 5	**Bb 4**	G 4	E 4	**D 4**	C 4

Table 4: The notes corresponding to the ratios in Table 3 when the root is C4..

For purposes of discussion, the reference note, or root, will be middle C denoted

[11] The Pythagoreans mixed math with religion, music and mysticism. They considered irrational numbers blasphemy. Singh [Simon Singh, *Fermat's Enigma* (Walker & Co., 1997)] tells of a Pythagorean heretic who brazenly – and correctly - suggested $\sqrt{2}$, like $\sqrt[12]{2}$, could not be expressed as the ratio of two whole numbers. This questioning of his teaching prompted Pythagoras to sentence the rebel to death by drowning.

C4. The first column of ratios in Table 3 are the integers 1 though 9. These are simply the harmonics of root. The first harmonic with a ratio 2.00 is one octave above C4 and is therefore C5. The second harmonic, with ratio 3.00, is G5, the perfect fifth of C5. The third harmonic with a ratio of 4.00 is one octave above C5 and is therefore C6. The fourth harmonic, as evidenced by the notes of *Taps*, is the perfect third of C6 and is therefore E6. The fifth harmonic, with ratio 6.00 is a factor of 2 greater than 3.00 and is therefore G6 – an octave above G5. The sixth harmonic is Bb6, the seventh harmonic, C7 and the eighth, equal to the perfect fifth of the fifth harmonic, is D7. These notes are entered into the ratio matrix in Table 4.

The entries in the top row of Table 3 can likewise be analyzed. The entries are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$. Rather than being sequential integer multiples of the root, the entries are sequential integer *divisors* of the root. These notes are *subharmonics*. With C4 as the root, the note with ratio $\frac{1}{2}$ is clearly an octave lower, or C3. The note with the ratio $\frac{1}{3}$ has C4 as its third harmonic. This note is F2. C2, an octave below C3, has a ratio of $\frac{1}{4}$. The ratio $\frac{1}{5}$ is the note having C4 as its fifth harmonic. This is Ab1. The last entry, with a value $\frac{1}{6}$, is an octave below $\frac{1}{3}$ and is therefore is F1.

Note	ratio	ratio	cents
Ab 3	(4,5)	(7,9)	-49
A 3	(5,6)	(6,7)	49
Bb 4	(7,4)	(9,5)	49
F# 3	(5,7)	(7,10)	-35
F# 4	(7,5)	(10,7)	35
Bb 3	(7,8)	(8,9)	27
D 4	(9,8)	(10,9)	-22

Table 5

Each of the columns in Figure 4 can now be filled. For purposes of discussion, refer to the *n*th row and the *m*th column in Tables 3 and 4 as (*n,m*) The intervals between two columns in the same row are preserved when the row is changed. In row one of Table 4, for example, the interval between (1,1) and (1,6) is a perfect fourth minus three octaves, i.e. C4 to F1. For any row, the entries in column 1 and 6 must be this same interval. Since the entry in row (7,1) has been determined to be Bb6, we remove three octaves and add a perfect fourth as we did in row 1, and the entry in (7,6) is Eb4. We proceed thus to fill in the entire matrix in Table 4.

There are problems with the natural Pythagorean ratios in Table 4 when the integers become intermediately large. The entries in Table 4 for (5,6) and (6,7) are both A3. The entry in Table 3 for (5,6) is 0.83 and the entry for (6,7) is 0.86. Different frequencies, separated by 49 cents or almost a half semitone, are assigned the same note: A3. Other inconsistencies are noted in Table 5. These entries include all ambiguities when ratios are taken between the

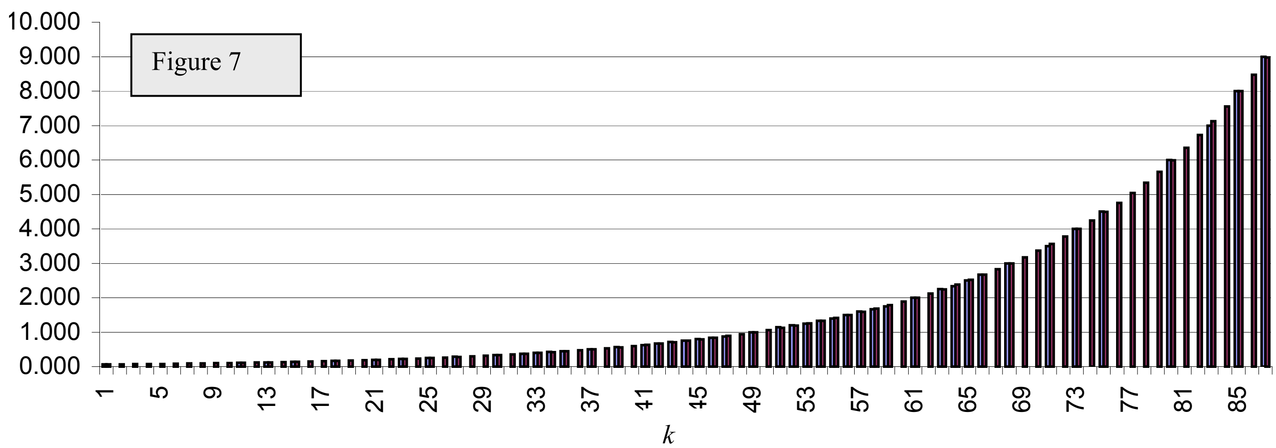
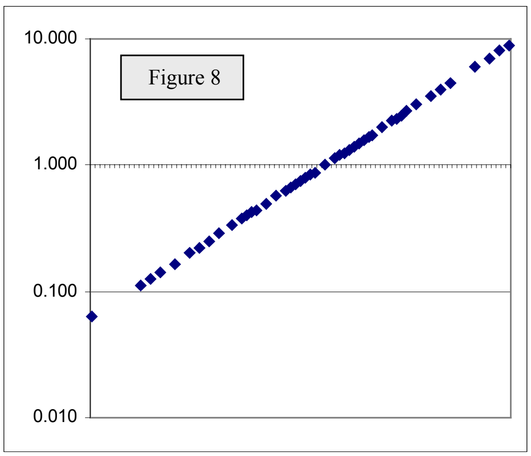


Figure 7

numbers one and ten. (Tables 3 and 4 are for one through nine).

The notes in Table 4 can be numbered starting at a value of 0 for the note C0. The number of semitones between C0 and is k. Each of the notes in Table 4 has a value of k. The variable k is then associated with the corresponding ratio value in Table 3. The sorting of these ratios in accordance to k is shown in Table C in Appendix C. Plotting the natural ratio and the tempered values results in the plot in Figure 7. The plot, by design, is an exponential curve. A logarithmic plot of the ratios, shown in Figure 8, nearly forms a line. Thus, the musical intervals are linear when perceived by a logarithmic sensor such as the human ear.



The straight line in Figure 8 may be initially surprising. The ratio of small numbers when matched to an indexed note form a remarkably straight line when plotted logarithmically. From another perspective, the result is not surprising at all. The natural ratios were matched to notes with the closest tempered values. The natural ratios, then, were dispersed in a fashion that forced their relationship in Figure 8 to be nearly linear.

The error analysis of the natural ratio to tempered value deviation is of greater significance. If matches between ratios and tempered intervals were random, we would expect errors to be uniformly distributed between –50 cents and 50 cents. This probability model has a standard deviation of

$$\sigma = \frac{100}{\sqrt{12}} = 28.9 \text{ cents.} \qquad (3)$$

The sample variance, obtained by evaluating the square root of average error squared, is

$$\sigma_S = \sqrt{\frac{1}{N}\sum_{k=1}^{N} c_k^2} = 16.5 \text{ cents}$$

where c_k is the error, in cents, read from the N=47 error entries in Table C. The result is significantly smaller than would be expected by uniformly random error.

Harmonious tempered intervals include major and minor thirds and sixths, perfect fourths and fifths, and octaves. If only the 30 entries corresponding to these harmonious intervals in Table C are used, the error standard deviation slips even lower to $\sigma_S = 12.1$.

Our experiment reveals, as expected, harmonious Pythagorean intervals obtained using a literal interpretation of Pythagoras' conjecture concerning ratios of small whole numbers yields notes significantly closer to tempered tones than random occurrence would otherwise allow.

Harmonics Expansions Produce Major Chords and the Major Scale

A more structured illustration of the tempered scale's ability to provide Pythagorean harmony comes from derivation of the tones in a tempered scale. We begin with the I, IV and V chords of the major scale. If C is the root, then I is a C major chord, IV is an F chord and V a G. Note G is the closest non-octave harmonic to C. In a dual sense, F has, as its closest non-octave harmonic, C. As before, all frequencies are normalized to the frequency of the root, C. The note C therefore has a normalized frequency of one.

In the construction of the I, IV, V chords, any note's frequency can be multiplied by 2^M without changing the note name as long as M is an integer. For any note, a value of M can always be found to place the notes normalized frequency between 1 and 2. Both 1 and 2 correspond to C's. Placing a note's normalized frequency between 1 and 2 is therefore simply constraining the note to lie in a specified octave.

The C major chord is formed from the first five harmonics of C. Removing the octave harmonics (the first and third), the notes of the C chord are C, G, and E with relative frequencies 1, 3, and 5. The frequency 3 for can be reduced to the desired octave by dividing by 2 to form the normalized frequency 3/2. Similarly, the frequency 5 can be divided by $4 = 2^2$ to form the normalized frequency 5/4. When reduced to octave between 1 and 2, The notes C, G and E have normalized frequencies of 1, 3/2, and 5/4.

A similar analysis is applicable to the V (G) chord. Beginning with a chord root of 3, the notes in the G chord, corresponding to the second and fourth harmonics, are 9 and 15. Thus, the G, D, and B notes of the G chord have respective frequencies 3, 9 and 15. By choosing an appropriate integer, M, each can be placed in the interval of 1 to 2. The 3, as before, is divided by 2 corresponding to $M = -1$. Using M=-3, the frequency of 9 becomes 9/8. Lastly, B becomes 15/8. The frequencies of the V chord in the frequency interval of 1 to 2 are therefore 3/2, 9/8 and 15/8.

Lastly, consider the IV chord. If we construe F as being the note to which C is the second harmonic, then frequency of the F note is 1/3. The notes for the F major chord, F, A, and C, follow as 1/3, 3/3 and 5/3. Each of these can be placed the interval 1 to 2. The result is 4/3, 1, and 5/3.

The results of our analysis are shown in Table 6. The top table has the harmonic construction of the I, IV, and V chords using harmonics followed by reducing the normalized frequencies to the octave between 1 and 2. The middle table contains the corresponding notes when the root note is a C. The last table is a decimal equivalent of the ratio in the top table.

ratios	IV F	I C	V G
Tonic	4/3	1	3/2
P 5th	1	3/2	9/8
M 3rd	5/3	5/4	15/8

notes	IV F	I C	V G
Tonic	F	C	G
P 5th	C	G	D
M 3rd	A	E	B

decimal	IV F	I C	V G
Tonic	1.3333	1.0000	1.5000
P 5th	1.0000	1.5000	1.1250
M 3rd	1.6667	1.2500	1.8750

Table 6: Constructing the Pythagorean 8 note major scale.

note	k	ratio	tempered	cents
C	0	1.0000	1.0000	0
C#	1		1.0595	
D	2	1.1250	1.1225	4
Eb	3		1.1892	
E	4	1.2500	1.2599	-14
F	5	1.3333	1.3348	-2
F#	6		1.4142	
G	7	1.5000	1.4983	2
Ab	8		1.5874	
A	9	1.6667	1.6818	-16
Bb	10		1.7818	
B	11	1.8750	1.8877	-12
C	12	2.0000	2.0000	0

Table 7: Comparison of the natural Pythagorean major scale with the tempered major scale.

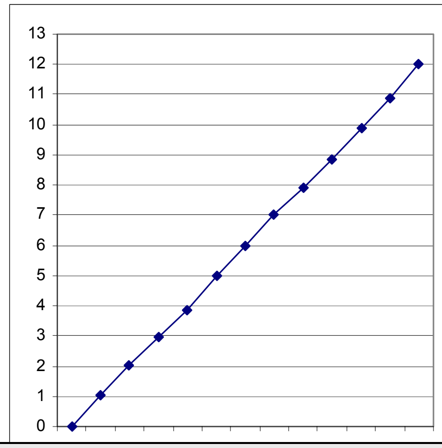


Figure 9: A logarithmic plot of the Pythagorean ratios in Table 7. The line is nearly straight illustrating the near equivalence of the harmonically derviced major scale with tempered intervals. If the line were exactly straight, each mark would lie on a horizontal line. (Missing entries in Table 7 were replaced by the geometric mean of ajdacent entries. The geometric mean of numbers p and q is $\sqrt{pq}$.

As before, we assign an index of k=0 to C, k=1 to C#, k=2 to D, etc. Arranging the entries in Table in ascending values of k gives the results shown in Table 7. A logarithmic plot of the ratio values, shown in Figure 9, results in a remarkably straight line. Using the entries in Table 7, the standard deviation of the natural 8 note major scale from the tempered scale is a miniscule 8.6 cents.[12] The error range (maximum error minus minimum error) in Table 7 is 20 cents compared to 100 cents expected for randomly chosen notes on the interval from 1 to 2.

ratios	F	C	G
root	4/3	1	3/2
3rd sub	16/9	4/3	1
5th sub	16/15	8/5	6/5

noted	Bbm	Fm	Cm
root	F	C	G
3rd sub	Bb	F	C
5th sub	Db	Ab	Eb

decimal	Bbm	Fm	Cm
root	1.3333	1.0000	1.5000
3rd sub	1.7778	1.3333	1.0000
5th sub	1.0667	1.6000	1.2000

Table 7: Chord expansion using subharmonics. The result is the notes in the key of F minor. The chords are I, IV and V chords of the key of (natural) F minor.

Consider, then, performing an operation similar to that in Table 6, except with subharmonics. The three notes, F, C, and G, will be expanded into their first four subharmonics. For each subharmonic, the note is moved into the octave with normalized frequencies between 1 and 2. The result is shown in Table 7. Interestingly, the notes generated are those in the natural minor key of Fm. The I, IV and V chords of the key are generated by the subharmonics.

Combining the Harmonic and Subharmonic Expansions Approximates the Tempered Chromatic Scale

Note that nearly all of the missing note in the harmonic expansion in Tables 6 and 7 are present in subharmonic expansion in Table 8. The only missing note in the union of the Tables is the tritone, F#. A logarithmic plot of these entries is shown in Figure 10. The standard deviation of the fit is a mere 12.1 cents.

[12] Recall, from Equation 3, the standard deviation of randomly placed notes is a significantly larger 28.9 cents.

Subharmonic Expansions Produce Minor Scales and Chords

The expansion in Table 6 used harmonics. A similar expansion can be performed using *subharmonics*. The *n*th subharmonic of a note has a frequency of 1/*n*th the root. Rather than being sequential integer multiples of the root as is the case with harmonic, subharmonics are sequential integer *divisors* of the root.

note	k	ratio	tempered	cents
C	0	1.0000	1.0000	0
C#	1	1.0667	1.0595	12
D	2	1.1250	1.1225	4
Eb	3	1.2000	1.1892	16
E	4	1.2500	1.2599	-14
F	5	1.3333	1.3348	-2
F#	6	x	1.4142	x
G	7	1.5000	1.4983	2
Ab	8	1.6000	1.5874	14
A	9	1.6667	1.6818	-16
Bb	10	1.7778	1.7818	-4
B	11	1.8750	1.8877	-12
C	12	2.0000	2.0000	0

Table 8: Eleven of the 12 tones of the tempered chromatic scale result naturally from harmonic and subharmonic expansion of the root, perfect fifth and perfect fourth. These entries are a combination of the entries in Table 5 (the harmonic expansion) and Table 7 (the subharmonic expansion). The natural Pythagorian results, as witnessed by the low cents error, are remarkably close to the corresponding tempered frequencies.

Wherein harmonics build major chords, subharmonics build minor chords. With C4 as the root, the note with ratio $\frac{1}{2}$ is clearly an octave lower, or C3. The note with the ratio $\frac{1}{3}$ has C4 as its third harmonic. This note is F2. C2, an octave below C3, has a ratio of $\frac{1}{4}$. The ratio $\frac{1}{5}$ is the note having C4 as its fifth harmonic. This is Ab1. The next subharmonic, with a value $\frac{1}{6}$, is an octave below $\frac{1}{3}$ and is therefore is F1. The notes generated by this sequence are C, F, and Ab. These notes constitute an F minor (written Fm) chord. Subharmonics, indeed, generate minor chords.

Final Remarks

A review of the discussion reveals, surprisingly, that no reason is given for the wonderfully close relationships of the natural harmonies of Pythagoras and those available from the tempered scale. No answer is given because none seemingly exists. Reasoning of the type given empirically establishes the relationship. The "why", however, remains mathematically and physically unanswered.

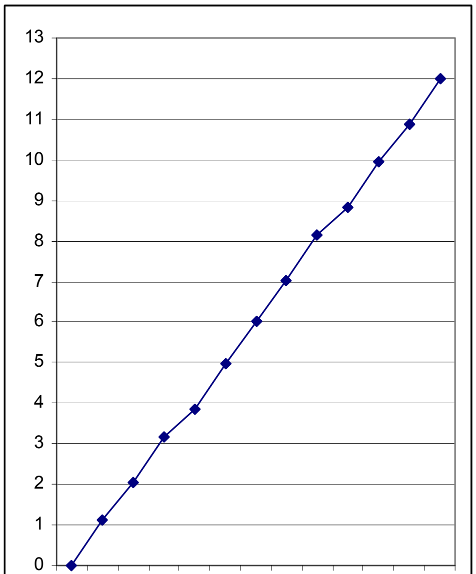


Figure 10: A logarithmic plot of chromatic intervals in Table 8 as predicted by harmonic and subharmonic expansions within an octave. Ideally, the plot should be a line with every tick exactly intersecting the horizontal line. Although not perfect, the naturally generated notes are nearly identical to their tempered equivalents. (The tritone was not generated in the harmonic and subharmonic expansions. The geometric mean was used to interpolate.)

Appendix A: The Wave Equation and its Solution

The wave equation is manifest in analysis of physical phenomena that display wave like properties. This includes electromagnetic waves, heat waves, and acoustic waves. We consider the case of the simple vibrating string.

Derivation

A string under horizontal tension T is subjected to a small vertical displacement, $y = y(x,t)$, that is a function of time, t, and location, x. As illustrated in Figure A1, attention is focused on an incremental piece of the string from x to $x+\Delta x$. Under the small displacement assumption, there is no movement of the string horizontally (*i.e.* in the x direction), and the horizontal forces must sum to zero.

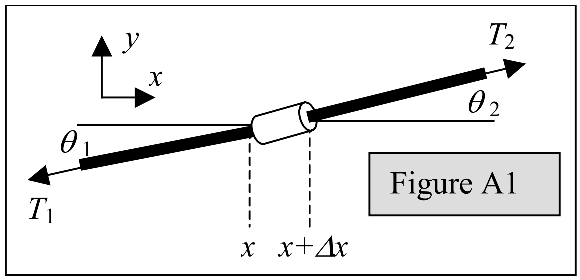


Figure A1

$$T = T_1 \cos\theta_1 = T_2 \cos\theta_2.$$

Let the linear mass density (*i.e.* mass per unit length) of the string be ρ. The mass of the incremental piece of string is then $\rho\Delta x$. The total vertical force acting on the string is $T_2 \cos\theta_2 - T_1 \cos\theta_1$. Using Newton's second law[13], we have

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = \rho\Delta x \frac{\partial^2 y}{\partial t^2}$$

Dividing by the x force equation gives

$$\frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} = \frac{\rho\Delta x}{T}\frac{\partial^2 y}{\partial t^2}$$

or

$$\tan\theta_2 - \tan\theta_1 = \frac{\rho\,\Delta x}{T}\frac{\partial^2 y}{\partial t^2}.$$

But

[13] Often written $F = ma$ or force equals mass times acceleration.

$$\tan\theta_2 = \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}$$

and

$$\tan\theta_1 = \left.\frac{\partial y}{\partial x}\right|_{x}$$

We can therefore write

$$\frac{1}{\Delta}\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

Taking the limit as $\Delta x \to 0$ and applying the definition of the derivative, we arrive at the wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad \text{(A1)}$$

where $v^2 = T / \rho$ is the velocity of the wave.

Solution

For a vibrating string fixed at x=0 and x=L, there are the boundary conditions

$$y(0,t) = y(L,t) = 0 \qquad \text{(A2)}$$

A solution satisfying the boundary conditions and the wave equation in (A1) is

$$y(x,t) = \sum_{m=1}^{\infty} A_m \cos\left(\frac{\pi m v t}{L}\right)\sin\left(\frac{\pi m x}{L}\right); 0 \le x \le L \qquad \text{(A3)}$$

where the A_m coefficients are to determined. This equation can be straightforwardly shown, by direct substitution, to satisfy both (A1) and (A2).

The coefficients, A_m , are determined by the initial displacement of the string

$$y(x,0) = f(x)$$

where $f(x)$ is any function satisfying the boundary conditions, $f(0) = f(L) = 0$. From A3,

$$f(x) = y(x,0) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{\pi m x}{L}\right); 0 \le x \le L$$

This expression is recognized as a Fourier series. The coefficients can be determined by

$$A_m = \frac{2}{L}\int_{x=0}^{L} f(x)\sin\left(\frac{\pi m x}{L}\right)dx$$

The Mth harmonic in (A3) is m+1 which, from the $\cos\left(\frac{\pi m v t}{L}\right)$ term in (A3), vibrates at a frequency of

$$f_m = \frac{mv}{2L}.$$

The root frequency is $f = f_1 = \frac{v}{2L}$ so

$$f_m = mf.$$

When m is fixed, the $\sin\left(\frac{\pi m x}{L}\right)$ term in (A3) has m "humps" on the interval $0 \le x \le L$. Thus, with reference to Figure 5, the M=1st harmonic has m=2 humps. The 2nd harmonic has m=3 humps, etc. The length of a hump is $\ell_m = L / m$. Thus, as advertised in (3),

$$\ell_m f_m = \left(\frac{L}{m}\right)\left(\frac{mv}{2L}\right) = v.$$

Appendix B: Computing the Semitones or Cents Interval Separating Two Frequencies

The number of semitones, n, a frequency f_A is above a reference frequency f_B, is

$$n = \log_{\sqrt[12]{2}}\left(\frac{f_B}{f_A}\right) \qquad \text{(B1)}$$

where $\log_{\sqrt[12]{2}}(\cdot)$ denotes a logarithm using $\sqrt[12]{2}$ as a base. If $f_A < f_B$, the value of n will be negative indicating the number of semitones f_A is below f_B.

This formula is equivalent to the more calculator friendly relationship.

$$n = \frac{\log\left(\frac{f_B}{f_A}\right)}{\log\left(\sqrt[12]{2}\right)} = \frac{12\log\left(\frac{f_B}{f_A}\right)}{\log(2)}$$

where the log can be taken with any base (*e.g.* 10 or *e*).

To illustrate, the frequency ratio of 3 to 2 is the natural Pythagorean perfect fifth. Substituting gives

$$n = \frac{12\log\left(\frac{3}{2}\right)}{\log(2)} = 7.01955.$$

The tempered perfect fifth is n = seven semitones above the root, so the near equivalence of the natural Pythagorean and the tempered interval is again established.

There are 100 cents in a semitone and 1200 cents in an octave. The number of cents, c, a frequency f_A is above a reference frequency f_B is

$$c = \log_{\sqrt[1200]{2}}\left(\frac{f_B}{f_A}\right) \qquad \text{(B1)}$$

where $\log_{\sqrt[1200]{2}}(\cdot)$ denotes a logarithm using $\sqrt[1200]{2}$ = the twelve hundredth root of two = 1.0005777895066, as its base. A more computational friendly formula is

$$c = \frac{1200\log\left(\frac{f_B}{f_A}\right)}{\log(2)}$$

where, as before, the log is any base.

For a given number of semitone intervals, n, the number of cents is

$$c = 100\, n.$$

Appendix C: Table C

Note	#	ratio	temper	cents
C 0	0	0.063	0.063	0
C#	1		0.066	
D	2		0.070	
Eb	3		0.074	
E	4		0.079	
F	5		0.083	
F#	6		0.088	
G	7		0.094	
Ab	8		0.099	
A	9		0.105	
Bb	10	0.111	0.111	4
B	11		0.118	
C 1	12	0.125	0.125	0
C#	13		0.132	
D	14	0.143	0.140	-31
Eb	15		0.149	
E	16		0.157	
F	17	0.167	0.167	2
F#	18		0.177	
G	19		0.187	
Ab	20	0.200	0.198	-14
A	21		0.210	
Bb	22	0.222	0.223	4
B	23		0.236	
C 2	24	0.250	0.250	0
C#	25		0.265	
D	26	0.286	0.281	-31
Eb	27		0.297	
E	28		0.315	
F	29	0.333	0.334	2
F#	30		0.354	
G	31	0.375	0.375	-2
Ab	32	0.400	0.397	-14
A	33	0.429	0.420	-33
Bb	34	0.444	0.445	4
B	35		0.472	
C 3	36	0.500	0.500	0
C#	37		0.530	
D	38	0.571	0.561	-31
Eb	39		0.595	
E	40	0.625	0.630	14
F	41	0.667	0.667	2
F#	42	0.714	0.707	-17
G	43	0.750	0.749	-2
Ab	44	0.800	0.794	-14
A	45	0.833	0.841	16
Bb	46	0.875	0.891	31
B	47		0.944	
C 4	48	1.000	1.000	0
C#	49		1.059	
D	50	1.143	1.122	-31
Eb	51	1.200	1.189	-16
E	52	1.250	1.260	14
F	53	1.333	1.335	2
F#	54	1.400	1.414	17
G	55	1.500	1.498	-2
Ab	56	1.600	1.587	-14
A	57	1.667	1.682	16
Bb	58	1.750	1.782	31
B	59		1.888	
C 5	60	2.000	2.000	0
C#	61		2.119	
D	62	2.250	2.245	-4
Eb	63	2.333	2.378	33
E	64	2.500	2.520	14
F	65	2.667	2.670	2
F#	66		2.828	
G	67	3.000	2.997	-2
Ab	68		3.175	
A	69		3.364	
Bb	70	3.500	3.564	31
B	71		3.775	
C 6	72	4.000	4.000	0
C#	73		4.238	
D	74	4.500	4.490	-4
Eb	75		4.757	
E	76		5.040	
F	77		5.339	
F#	78		5.657	
G	79	6.000	5.993	-2
Ab	80		6.350	
A	81		6.727	
Bb	82	7.000	7.127	31
B	83		7.551	
C 7	84	8.000	8.000	0
C#	85		8.476	
D	86	9.000	8.980	-4
Eb	87		9.514	
E	88		10.079	
F	89		10.679	
F#	90		11.314	
G	91		11.986	
Ab	92		12.699	
A	93		13.454	
Bb	94		14.254	
B	95		15.102	
C 8	96	16.000	16.000	0